Student seminar exercise sheet Week 14

Let F be a number field.

1. Let

$$\begin{array}{rccc} \Phi: & \{\text{finite abelian extension } K \text{ of } F\} & \longrightarrow & \{\text{open subgroups } \mathcal{H} \text{ of } J_F \text{ that contains } F^\times\} \\ & K & \longmapsto & F^\times N_{K/F}\, J_F \end{array}$$

   Let K, K' be finite abelian field extensions of F. Show that Φ has the following properties

   iii. $\Phi(K \cap K') = \Phi(K)\Phi(K')$.

   iv. If $\mathcal{H} = \Phi(E) = F^\times N_{E/F} J_E$ and $K \supseteq E$, then E is the fixed field of $\rho_{K/F}(\mathcal{H})$.

2. The group of $\mathcal{S}$-idèle classes of F is

$$C_{F,\mathcal{S}} = {C_F}/{F_{\mathcal{S}}}.$$

   We have $J_{F,\mathcal{S}} \subseteq J_F$ and $F_{\mathcal{S}} \subseteq F^\times$, so there a natural embedding $C_{F,\mathcal{S}} \hookrightarrow C_F$. Show that there is a topological and algebraic isomorphism

$${J_F}/{F^\times J_{F,\mathcal{S}}} \cong {C_F}/{C_{F,\mathcal{S}}}$$

3. Let $\mathcal{S} \subseteq V_F$ be a finite set of places of F, containing all infinite places.

   (a) Let $\mathcal{S}_0 = \mathcal{S} - \mathcal{S}_\infty$ and

$$\begin{array}{rccc} \gamma: & F_{\mathcal{S}} & \longrightarrow & \mathcal{I}_{\mathcal{S}_0} \\ & \alpha & \longmapsto & \langle \alpha \rangle \end{array}$$

   where $\mathcal{I}_{\mathcal{S}_0}$ is the group of fractional ideals of F generated by $\{\mathfrak{p}_v : v \in \mathcal{S}_0\}$. Show that there is an exact sequence

$$1 \longrightarrow \mathcal{O}_F^\times \longrightarrow F_{\mathcal{S}} \xrightarrow{\gamma} \mathcal{I}_{\mathcal{S}_0}\ .$$

   (b) Show that $\gamma(F_{\mathcal{S}})$ is of rank $\#\mathcal{S}_0$.

   (c) Show that $F_S \cong \mathcal{O}_F^\times \times \gamma(F_{\mathcal{S}})$.

   (d) Conclude that

$$F_{\mathcal{S}} \cong \mathcal{W}_F \times \mathbb{Z}^{\#\mathcal{S}-1}$$

   where $\mathcal{W}_F$ is the group of roots of unity in F.

4. We look at Kummer extension of some fields.

   (a) What are the Kummer 2-extension of $\mathbb{Q}$?

   (b) What if we add a third root of unity ? Namely, what are the Kummer 3-extension of $\mathbb{Q}(\zeta_3)$?

5. Let $n > 0$ and $\mathcal{S}$ be a finite set of places of F, where F contains all n-th root of unity. Moreover assume that for every prime $\mathfrak{p}_v$ dividing n, the associated finite place v belongs to $\mathcal{S}$ and that $\mathcal{S}$ contains all infinite places and is large enough so that $J_F = F^\times J_{F,\mathcal{S}}$. Finally let

$$B = \prod_{v \in \mathcal{S}} (F_v^\times)^n \times \prod_{v \notin \mathcal{S}} \mathcal{U}_v.$$

   Show that $B \cap F^\times = F_{\mathcal{S}}^n$.